Computational Modeling of the Cardiovascular System

Mechanical Modeling of Tissues I

CVRTI

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Overview

- Recapitulation Force Development
- Measurement System
- Definitions
- Strain/Stress Tensors
- Constitutive Laws
- Homework II

Group work & pause

Group work

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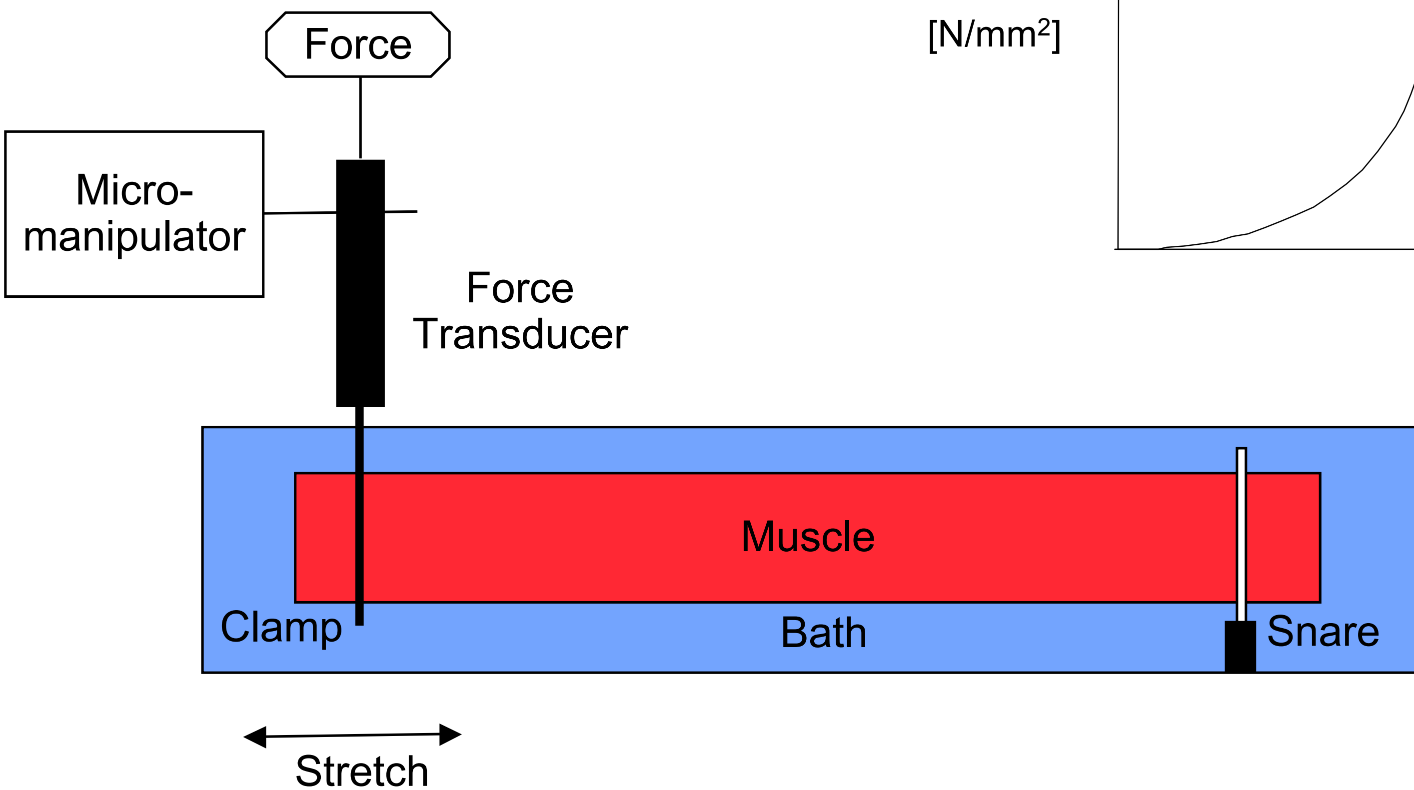
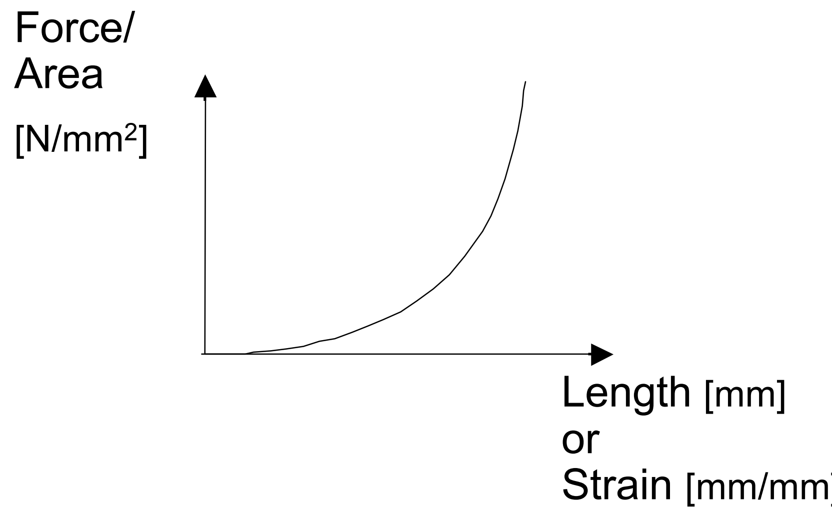
Uniaxial Measurement System





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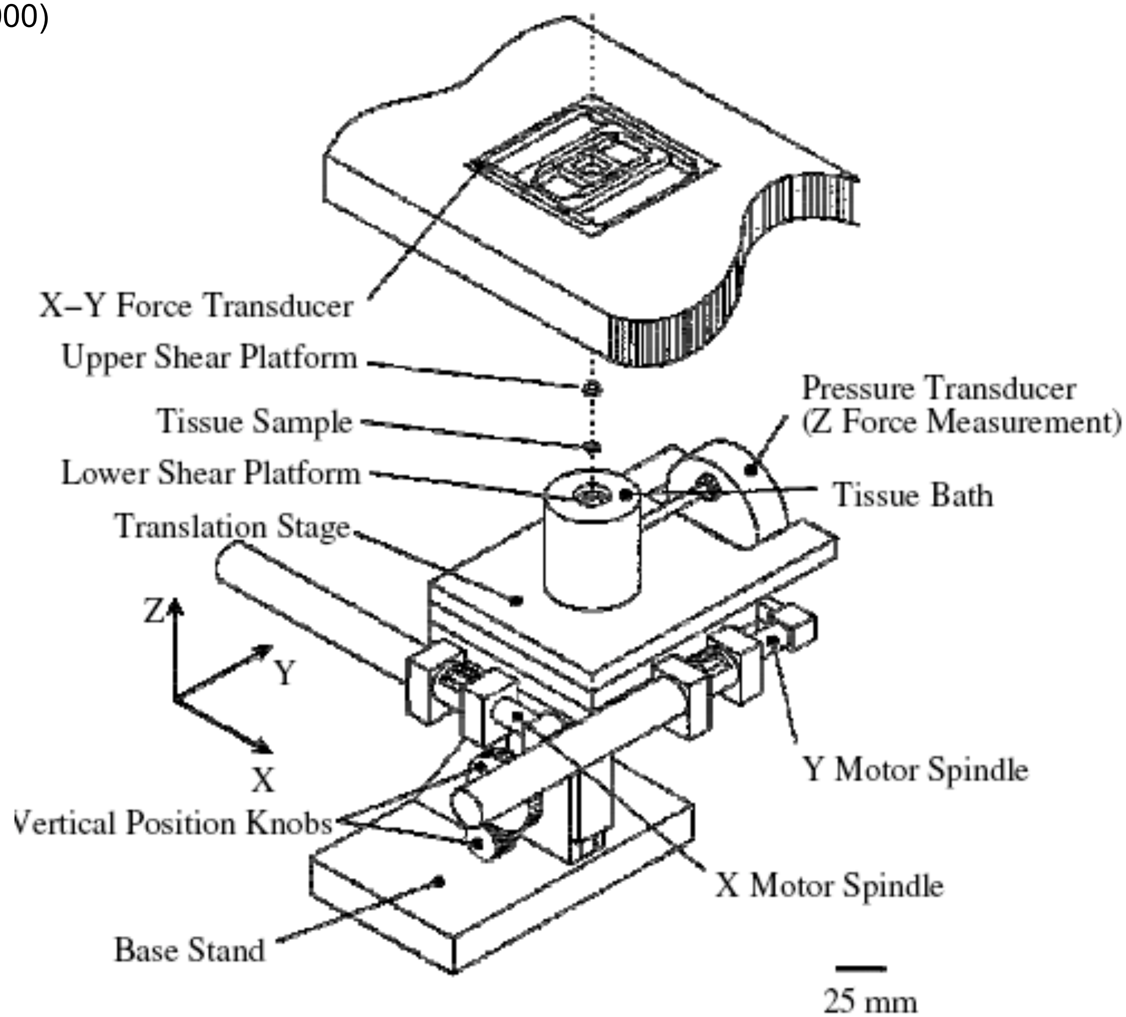
Triaxial Measurement System for Soft Biological Tissue

(Dokos et al., J Biomech Eng, 2000)



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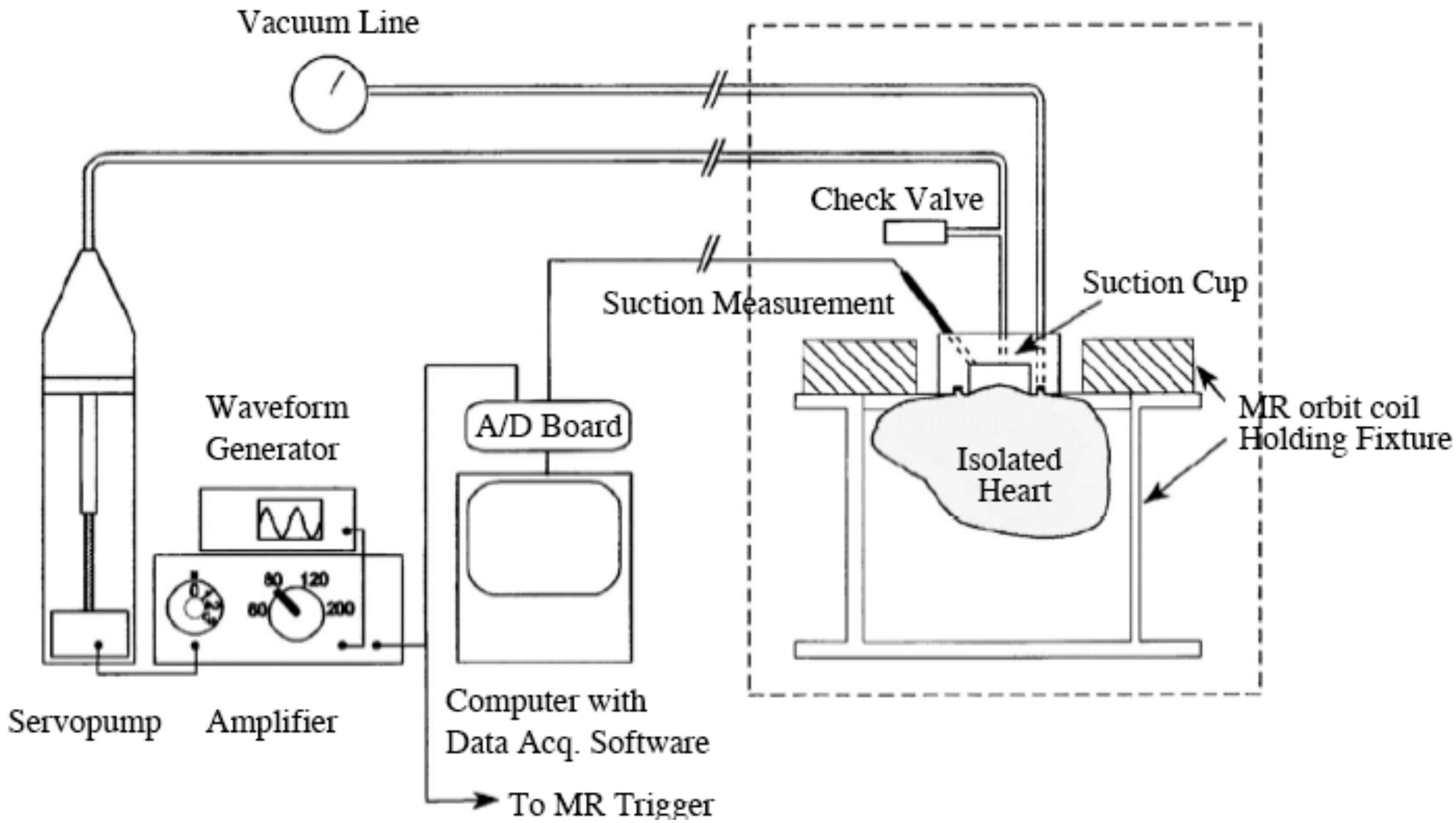
Epicardial Suction Measurement System



(Okamoto et al., J Biomech Eng, 2000)

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Biomechanics: Overview

Mathematical model

Tissue simulation

- **Kinematics**
  Description of motion and derived quantities, e.g.:
  Deformation gradient, strain, and velocity

- **Equilibrium of forces**
  Translational and rotational equilibrium
  Conservation of energy

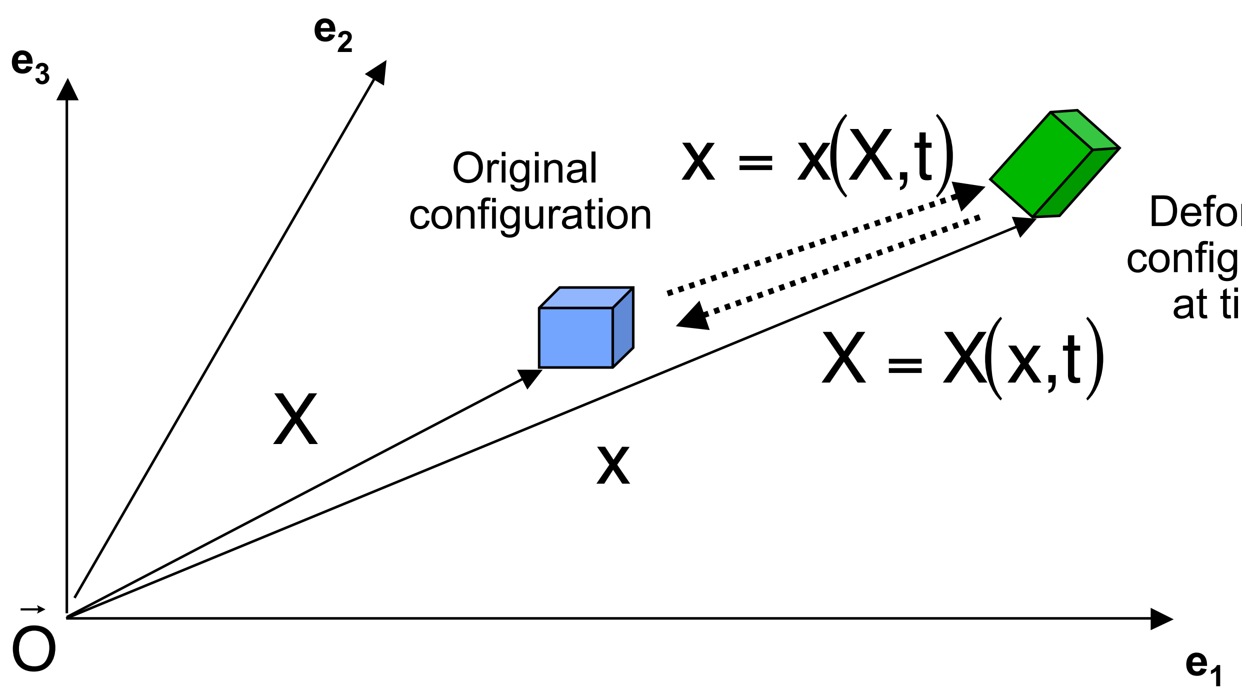
- **Constitutive relationships**
  Relationship between strain and stress
  Inclusion of measurement data

- **Boundary conditions**
  Restriction of motion
  Generally also forces, distribution of stresses, pressure etc.

Configurations of a Body

R_{t0} : Reference configuration
Lagrange configuration

R_t : Configuration at time t
Euler configuration



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Example for Mathematical Description of Motion

R_{t0} : Reference configuration

Lagrange configuration

$$R_{t0} \rightarrow R_t$$

$$x_1(X_1,X_2,X_3,t) = X_1 + X_2\left(e^t - 1\right)$$

$$x_2(X_1,X_2,X_3,t) = X_1\left(e^{-t} - 1\right) + X_2$$

$$x_3(X_1,X_2,X_3,t) = X_3$$

R_t : Configuration at time t

Euler configuration

$$R_t \rightarrow R_{t0}$$

$$X_1(x_1,x_2,x_3,t) = \frac{-x_1 + x_2\left(e^t - 1\right)}{1 - e^t - e^{-t}}$$

$$X_2(x_1,x_2,x_3,t) = \frac{x_1\left(e^{-t} - 1\right) - x_2}{1 - e^t - e^{-t}}$$

$$X_3(x_1,x_2,x_3,t) = x_3$$

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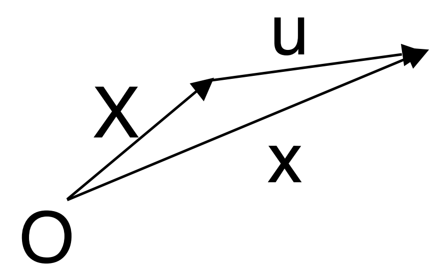
Displacement Vector

$$x(X,t) = X + u(X,t)$$

X : Reference coordinates of a point

$x(X,t)$: Coordinates of point at time t

$u(X,t)$: Displacement of point at time t



Example:

$$x_1(X_1,X_2,X_3,t) = X_1 + X_2\left(e^{t} - 1\right) \qquad u_1(X_1,X_2,X_3,t) = X_2\left(e^{t} - 1\right)$$

$$x_2(X_1,X_2,X_3,t) = X_1\left(e^{-t} - 1\right) + X_2 \qquad u_2(X_1,X_2,X_3,t) = X_1\left(e^{-t} - 1\right)$$

$$x_3(X_1,X_2,X_3,t) = X_3 \qquad u_3(X_1,X_2,X_3,t) = 0$$

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Description of Elementary Deformations

dX
X

$F = G^{-1}$

$dx = F\,dX$ $\qquad$ $dX = G\,dx$

dx
x

$x = x(X,t)$

"Derivative of Euler regarding Lagrange"

$$F = \left[\frac{\partial x_i}{\partial X_j}\right] = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix}$$

F: Deformation gradient tensor

$X = X(x,t)$

"Derivative of Lagrange regarding Euler"

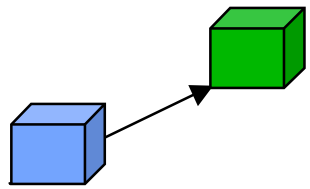
$$G = \left[\frac{\partial X_i}{\partial x_j}\right] = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{pmatrix}$$

G: Inverse deformation gradient tensor

Deformation Gradient for Translation/Scaling

Translation:

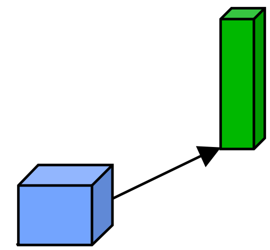
$$x_1 = X_1 + c_1$$
$$x_2 = X_2 + c_2$$
$$x_3 = X_3 + c_3$$



$$F = \left[\frac{\partial x_i}{\partial X_j}\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Scaling:

$$x_1 = \lambda_1 X_1$$
$$x_2 = \lambda_2 X_2$$
$$x_3 = \lambda_3 X_3$$



$$F = \left[\frac{\partial x_i}{\partial X_j}\right] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

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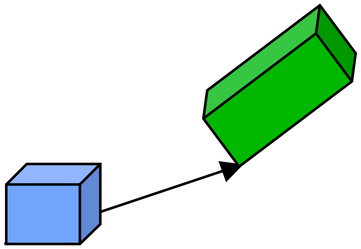
Deformation Gradient for Translation and Scaling

Scaling and rotation:

$$x_1 = X_1 - \lambda X_2$$
$$x_2 = \lambda X_1 + X_2$$
$$x_3 = X_3$$



$$F = \left[\frac{\partial x_i}{\partial X_j}\right] = \begin{pmatrix} 1 & -\lambda & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Deformation Gradient für Shearing

Shear:

$$x_1 = X_1 + \lambda X_2$$
$$x_2 = \lambda X_1 + X_2$$
$$x_3 = X_3$$

$$F = \left[\frac{\partial x_i}{\partial X_j}\right] = \begin{pmatrix} 1 & \lambda & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

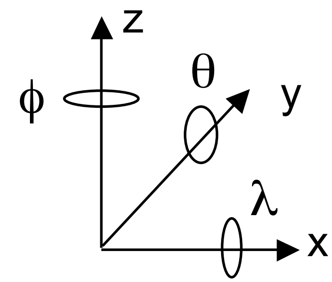
Simple Shear:

$$x_1 = X_1 + \lambda X_2$$
$$x_2 = X_2$$
$$x_3 = X_3$$

$$F = \left[\frac{\partial x_i}{\partial X_j}\right] = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Polar Decomposition of Deformation Gradient

$$F = RU \quad \begin{cases} R: & \text{Orthonormal rotation matrix with the angles } \phi, \theta, \lambda \\ U: & \text{Symmetric positive definite scaling matrix} \end{cases}$$



$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{pmatrix}$$

$$R = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda & \sin\lambda \\ 0 & -\sin\lambda & \cos\lambda \end{pmatrix}$$

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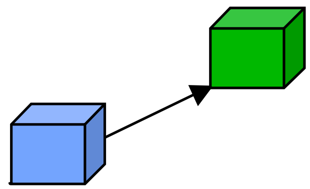
Deformation Gradient for Translation/Scaling

Translation:

$$x_1 = X_1 + c_1$$

$$x_2 = X_2 + c_2$$

$$x_3 = X_3 + c_3$$

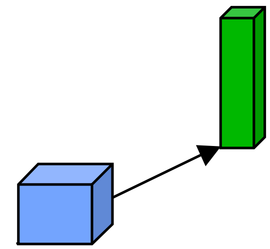


Scaling:

$$x_1 = \lambda_1 X_1$$

$$x_2 = \lambda_2 X_2$$

$$x_3 = \lambda_3 X_3$$



$$F = \left[\frac{\partial x_i}{\partial X_j}\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F = \left[\frac{\partial x_i}{\partial X_j}\right] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$F = RU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F = RU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

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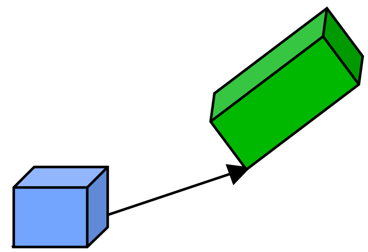
Deformation Gradient for Translation and Scaling

Scaling and rotation

$$x_1 = X_1 - \lambda X_2$$
$$x_2 = \lambda X_1 + X_2$$
$$x_3 = X_3$$



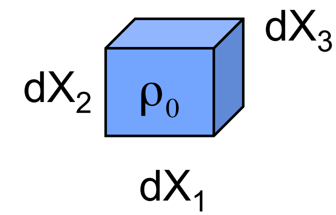
$$F = \left[\frac{\partial x_i}{\partial X_j}\right] = \begin{pmatrix} 1 & -\lambda & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F = RU = \begin{pmatrix} \cos\arctan\lambda & -\sin\arctan\lambda & 0 \\ \sin\arctan\lambda & \cos\arctan\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1+\lambda^2} & 0 & 0 \\ 0 & \sqrt{1+\lambda^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Change of Volume

$dV = dX_1 \, dX_2 \, dX_3$

dV: Volume in Lagrange configuration

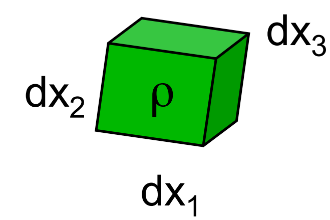


$dv = dx_1 \, dx_2 \, dx_3$

dv: Volume in Euler configuration

$dv = J dV$

J: Jacobian
Determinant of deformation gradient F



Conservation of volume: $J = 1$

Conservation of mass: $J = \frac{\rho_0}{\rho}$

ρ_0 : Density in Lagrange configuration ρ : Density in Euler configuration

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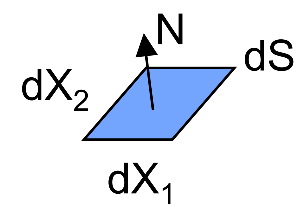
Change of Surface

$$n\,ds = dx_1 \times dx_2$$

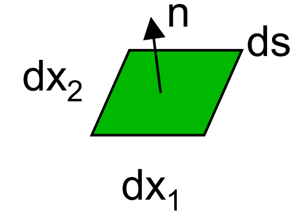
$$N\,dS = dX_1 \times dX_2$$



$$n\,ds = JG^T N\,dS$$

Lagrange configuration

dS: Surface area
N: Surface normal



Euler configuration

ds: Surface area
n: Surface normal

$$J = \det(F)$$

$$G = F^{-1}$$

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Analysis of Deformation: Strains

Relationship between elementary quadratic deformation in Lagrange and Euler configurations as symmetric tensors of 2nd order

$$dx^2 = dx^T dx = dX^T F^T F\, dX = dX^T C\, dX$$

$C = F^T F$: (Right) Cauchy-Green Deformation Tensor

$B = FF^T$ Left Cauchy-Green Deformation Tensor

$$dX^2 = dX^T dX = dx^T G^T G dx = dx^T c\, dx$$

$c = G^T G$: Cauchy Strain Tensor

$$c = G^T G = \left(F^{-1}\right)^T \left(F^{-1}\right) = \left(FF^T\right)^{-1} = B^{-1}$$

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Analysis of Deformation: Strains

Relationship between differences of elementary quadratic deformation in Lagrange and Euler configurations as symmetric tensors of 2nd order

$$dx^2 - dX^2 = dX^T C\, dX - dX^2 = dX^T (C - I) dX = dX^T 2E dX$$

$$E = \frac{1}{2}(C - I):$$ Lagrangian Strain Tensor

$$dx^2 - dX^2 = dx^2 - dx^T c\, dx = dx^T (I - c) dx = dx^T 2 e\, dx$$

$$e = \frac{1}{2}(I - c):$$ Euler-Almansi Strain Tensor

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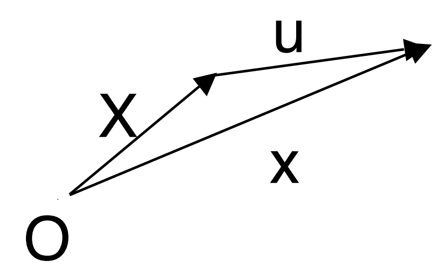
Analysis of Displacements: Strains

$u = x - X$ ← $f(x_1, x_2, x_3)$

u: Displacement

X: Lagrangian coordinates of point

x: Eulerian coordinates of point



$$\nabla x = F \quad \Rightarrow \quad \nabla u = F - I$$

$$du = dx - dX = (F - I)dX = (I - G)dx$$

$$du = \nabla u \, dX$$

Deformation gradient tensor: $F = I + \nabla u$

Cauchy-Green deformation tensor: $C = (I + \nabla u)^T (I + \nabla u) = I + \nabla u + \nabla u^T + \nabla u^T \nabla u$

Lagrangian strain tensor: $E = \frac{1}{2}\left(\nabla u + \nabla u^T + \nabla u^T \nabla u\right)$

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Linearization

Lagrangian strain tensor: $$E = \frac{1}{2}\left(\nabla u + \nabla u^T + \nabla u^T \nabla u\right)$$

$\overset{\text{small u}}{\Rightarrow}$ Infinitesimal Lagrangian strain tensor: $$\varepsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

(Symmetric 2nd order tensor)

$$\varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \text{ mit } \begin{cases} \varepsilon_{11} = \dfrac{\sigma u_1}{\partial X_1} & \varepsilon_{12} = \varepsilon_{21} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1}\right) \\ \varepsilon_{22} = \dfrac{\partial u_2}{\partial X_2} & \varepsilon_{13} = \varepsilon_{31} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3} + \dfrac{\partial u_3}{\partial X_1}\right) \\ \varepsilon_{33} = \dfrac{\partial u_3}{\partial X_3} & \varepsilon_{23} = \varepsilon_{32} = \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3} + \dfrac{\partial u_3}{\partial X_2}\right) \end{cases}$$

$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$: Strain in x-, y-, and z-direction

$\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$: Shear in xy-, xz-, and yz-plane

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Vectorial Representation of Strain Tensors

$$\varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \rightarrow \hat{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 2\varepsilon_{12} & 2\varepsilon_{13} & 2\varepsilon_{23} \end{pmatrix}^T$$

(classical strain tensor)

$$E = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{pmatrix} \rightarrow \hat{E} = \begin{pmatrix} E_{11} & E_{22} & E_{33} & 2E_{12} & 2E_{13} & 2E_{23} \end{pmatrix}^T$$

(Lagrangian strain tensor)

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Strain Tensors after Polar Decomposition

$$F = RU \quad \begin{cases} R\text{: Rotation tensor} \\ U\text{: Scaling tensor} \end{cases}$$

$$C = F^T F = (RU)^T (RU) = \left(U^T R^T\right)(RU) = U^2$$

$$E = \frac{1}{2}(C - I) = \frac{1}{2}\left(U^2 - I\right)$$

For rigid deformation: $\Rightarrow \begin{cases} U = I \\ F = R \\ C = B = c = I \\ E = e = \varepsilon = 0 \end{cases}$

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Tensors und Coordinate Transforms

Restriction to Cartesian three-dimensional tensors
(Cartesian: rectilinear coordinate system)

Transformation via deformation gradient F (here, orthogonal 3x3 matrix)

Order	Example	Transformation
0	Electrical voltage	$\xi' = \xi$
1	Electrical field	$\xi'_i = \sum_k F_{i,k} \xi_k$
2	Electrical conductivity	$\xi'_{i,j} = \sum_{k,l} F_{i,k} F_{j,l} \xi_{k,l}$
4	Material tensor in mechanics	$\xi'_{i,j,k,l} = \sum_{m,n,r,s} F_{i,m} F_{j,n} F_{k,r} F_{l,s} \xi_{m,n,r,s}$

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Invariants of Tensors

Invariants are independent of orthonormal tensor transformations

$$I_K = \mathrm{tr}(K) = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_K = \frac{1}{2}\left(\mathrm{tr}(K)^2 - \mathrm{tr}\left(K^2\right)\right) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

$$III_K = \det(K) = \lambda_1\lambda_2\lambda_3$$

$\mathrm{tr}(K) = K_{11} + K_{22} + K_{33}$ (Trace of a matrix)

$\lambda_1, \lambda_2, \lambda_3$: Eigenvalues of matrix K

Invariants of strain tensors are used to describe material isotropic properties

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Invariants of Cauchy-Green Deformation Tensor

$$I_C = I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$II_C = I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2$$

$$III_C = I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

C: Right Cauchy-Green deformation tensor with $C = F^2 = U^2$

$\lambda_1, \lambda_2, \lambda_3$: Eigenvalues of U

Invariants of strain tensors are used to describe material isotropic properties, e.g. for Mooney-Rivlin model of isotropic, rubber-like material

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Group Work

Specify 10 physical quantities and their tensor order(s)!

Why can we describe a physical quantity with tensors of different order?

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Analysis of Stress: Stress Vector

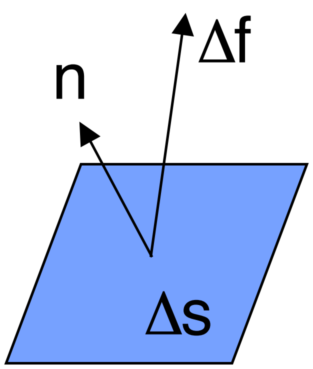
$$t^n = \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} = \frac{df}{ds}$$

Δf: Force vector $[N]$

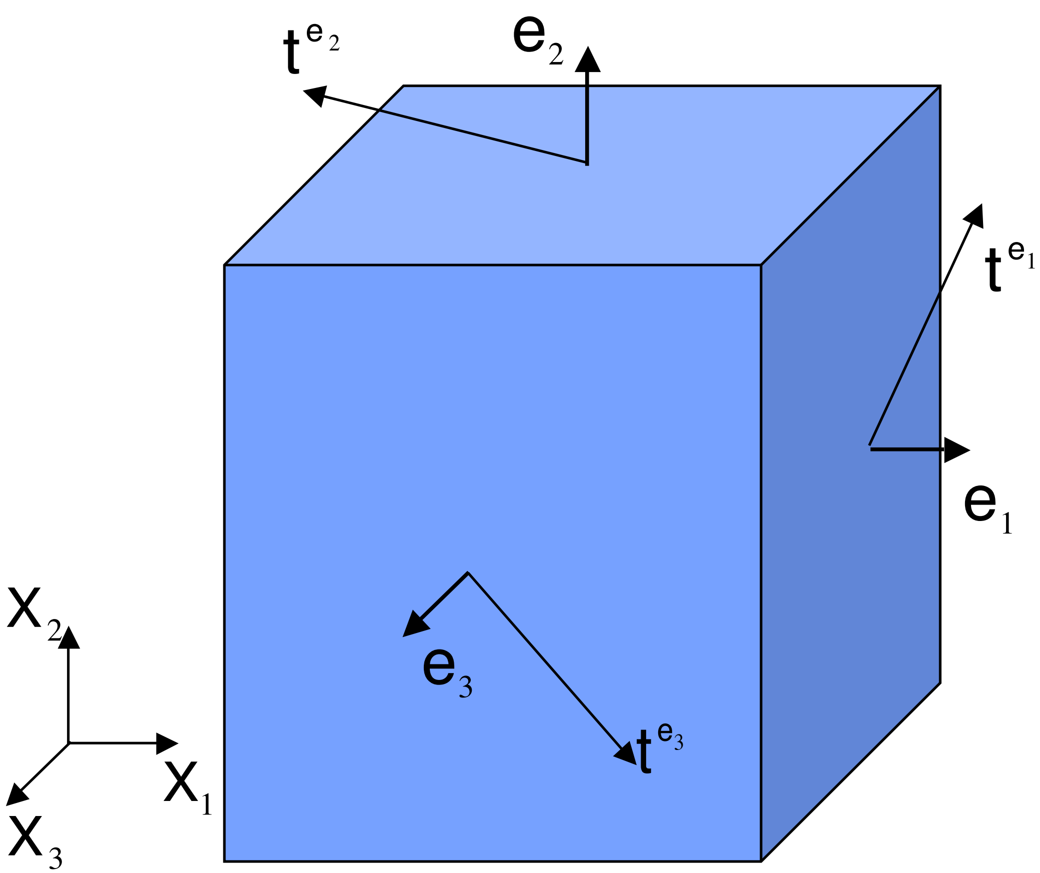
Δs: Area $[m^2]$

t^n: Stress vector $[N/m^2]$

n: Normal vector $[m]$



Analysis of Stress: Stress Vector



e_1: Unit vector in X_1-direction

$(1 \quad 0 \quad 0)^T$

e_2 Unit vector in X_2-direction

$(0 \quad 1 \quad 0)^T$

e_3 Unit vector in X_3-direction

$(0 \quad 0 \quad 1)^T$

t^{e_1}: Stress vector for X_{23}-plane

t^{e_2}: Stress vector for X_{13}-plane

t^{e_3}: Stress vector for X_{12}-plane

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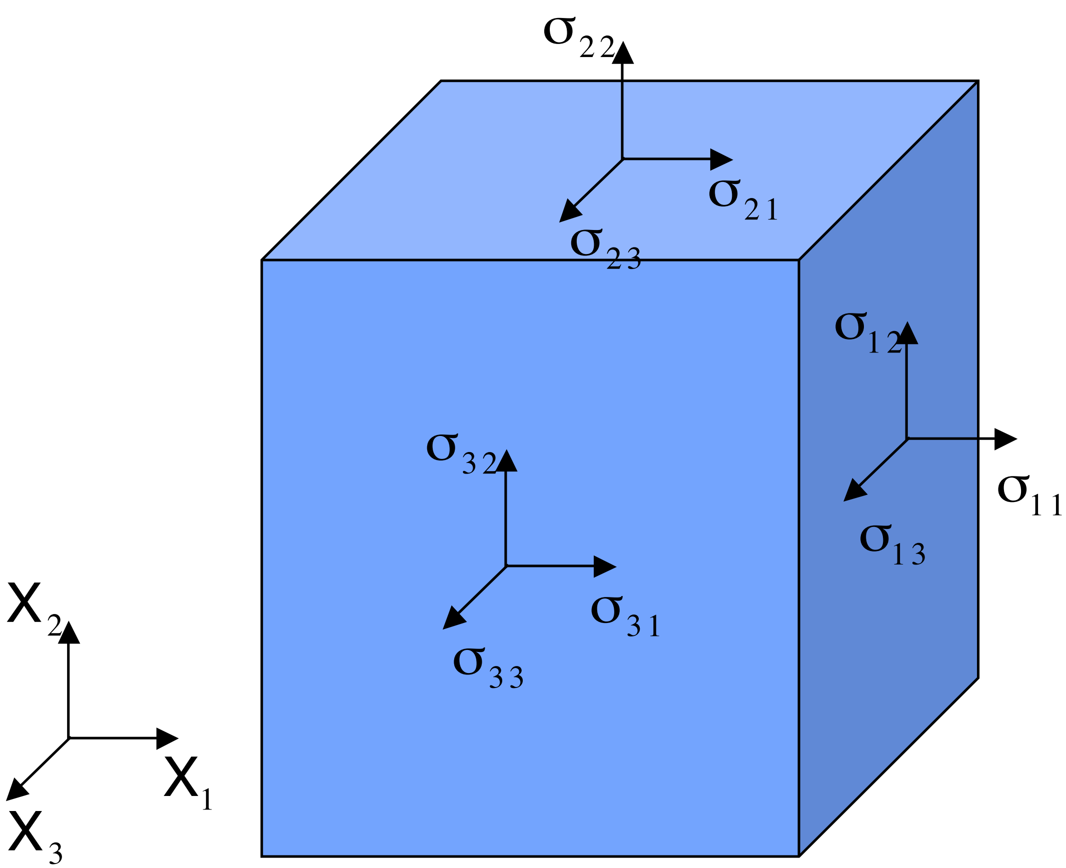
Analysis of Stress: Cauchy Stress Tensor



$$t^{e_i} = \left(t_1^{e_i} \quad t_2^{e_i} \quad t_3^{e_i}\right)^T$$

$$t_j^{e_i} = \sigma_{ij}$$

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Cauchy stress tensor

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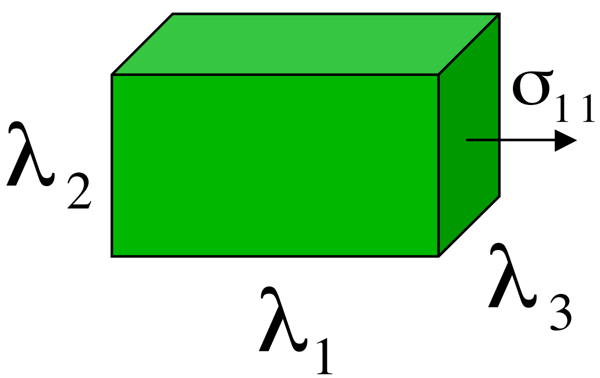
Cauchy Stress Tensor

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Stress in direction of normals

Shear stress

Tensor of 2. order, symmetric



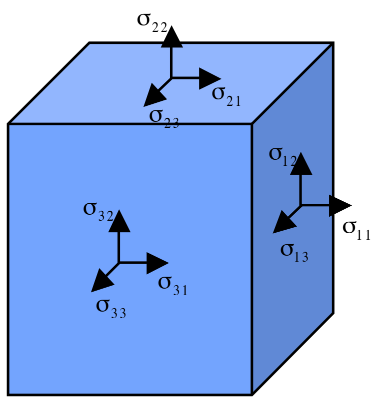
$$F_S = t^{e_1}\lambda_2\lambda_3 = \sigma e_1 \lambda_2 \lambda_3 = \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{pmatrix} \lambda_2 \lambda_3$$

Analysis of Stress: Example

Given for point P:

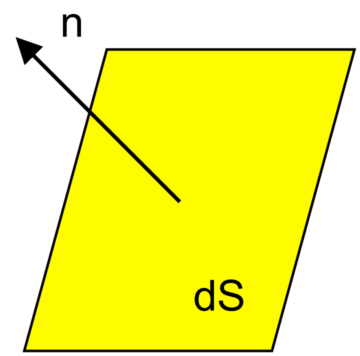
Cauchy stress tensor: $\sigma = \begin{pmatrix} 6 & 1 & -2 \\ 1 & 4 & 0 \\ -2 & 0 & 3 \end{pmatrix}$

Surface normal: $n = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}^T$



Wanted at point P:

Stress vector t^n



Solution:

$t^n = \sigma n$

...

Analysis of Stress: 1. Piola-Kirchhoff Stress Tensor

Given: Stress tensor σ in Euler configuration
Wanted: Stress tensor T in Lagrange configuration

Integral transformation (Surface change): $n\, dS_t = J G^T N\, dS_0$

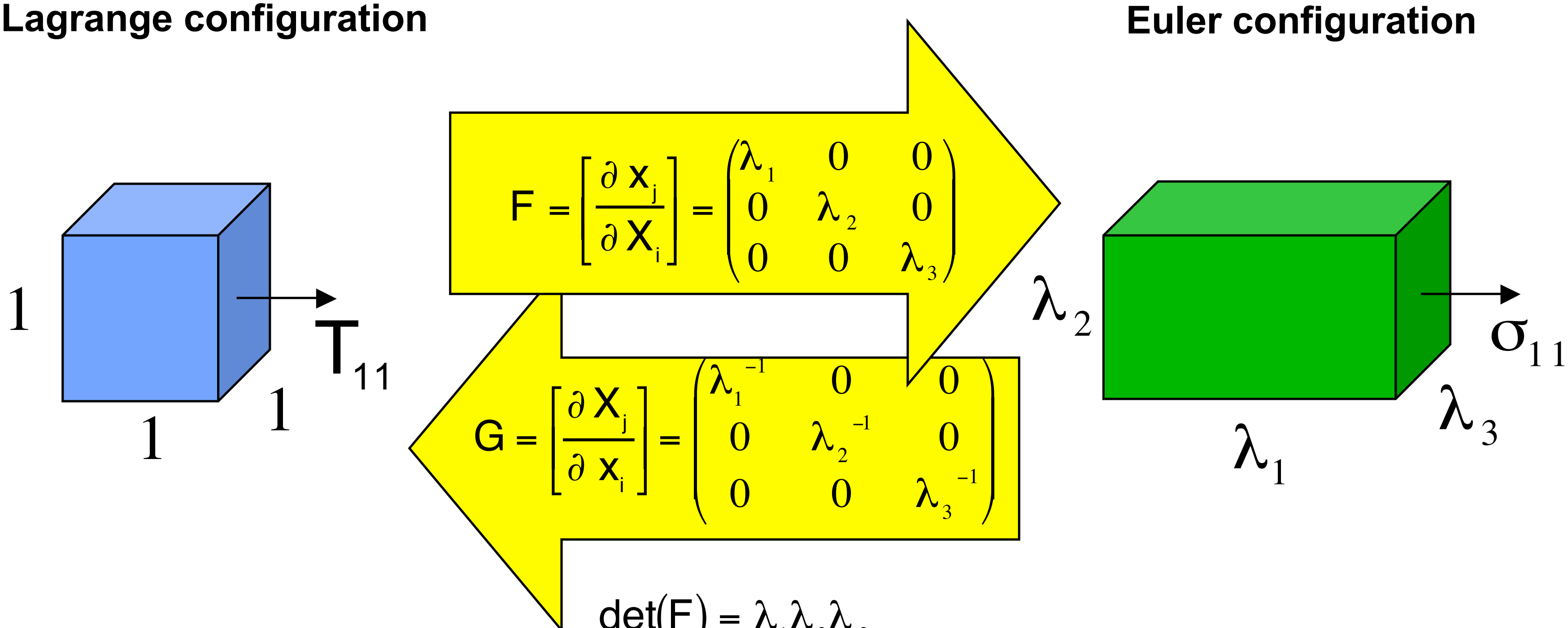
$$\int_{S_t} \sigma^T n\, dS_t = \int_{S_0} J \sigma^T G^T N\, dS_0 = \int_{S_0} T^T N\, dS_0$$

$T = JG\sigma$: 1. Piola-Kirchhoff Stress Tensor (Lagrange)
Tensor 2. order, asymmetric

CVRTI

Example: 1. Piola-Kirchhoff Stress Tensor

Lagrange configuration **Euler configuration**



$$F = \left[\frac{\partial x_j}{\partial X_i}\right] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$G = \left[\frac{\partial X_j}{\partial x_i}\right] = \begin{pmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{pmatrix}$$

$$\det(F) = \lambda_1 \lambda_2 \lambda_3$$

$$T_{11} = \lambda_2 \lambda_3 \sigma_{11}$$

CVRTI

Analysis of Stress: 2. Piola-Kirchhoff Stress Tensor

Given: Stress tensor σ in Euler configuration
Wanted: Stress tensor T in Lagrange configuration

Integral transformation (Surface change): $n\, dS_t = JG^T N\, dS_0$

$$\int_{S_t} \sigma^T n\, dS_t = \int_{S_0} J\sigma^T G^T N\, dS_0 = \int_{S_0} T^T N\, dS_0 = \int_{S_0} F S^T N\, dS_0$$

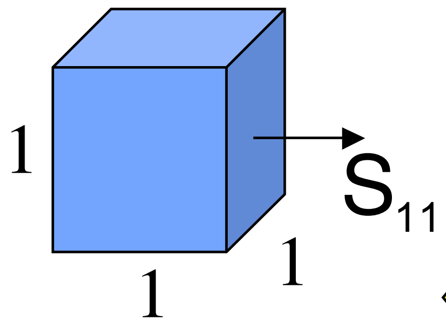
$S = TG^T = JG\sigma G^T$: 2. Piola-Kirchhoff Stress Tensor
Tensor 2. order, symmetric

CVRTI

Example: 2. Piola-Kirchhoff Stress Tensor

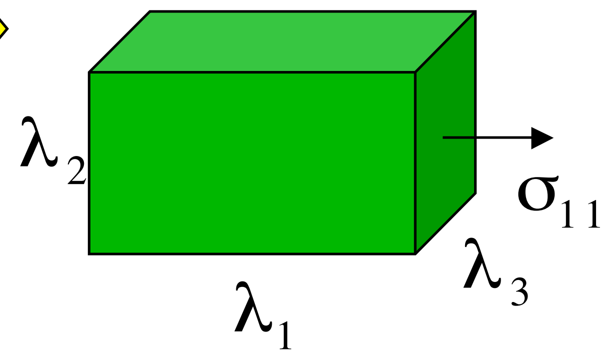
Lagrange configuration

Euler configuration



$$F = \left[\frac{\partial x_j}{\partial X_i}\right] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$G = \left[\frac{\partial X_j}{\partial x_i}\right] = \begin{pmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{pmatrix}$$

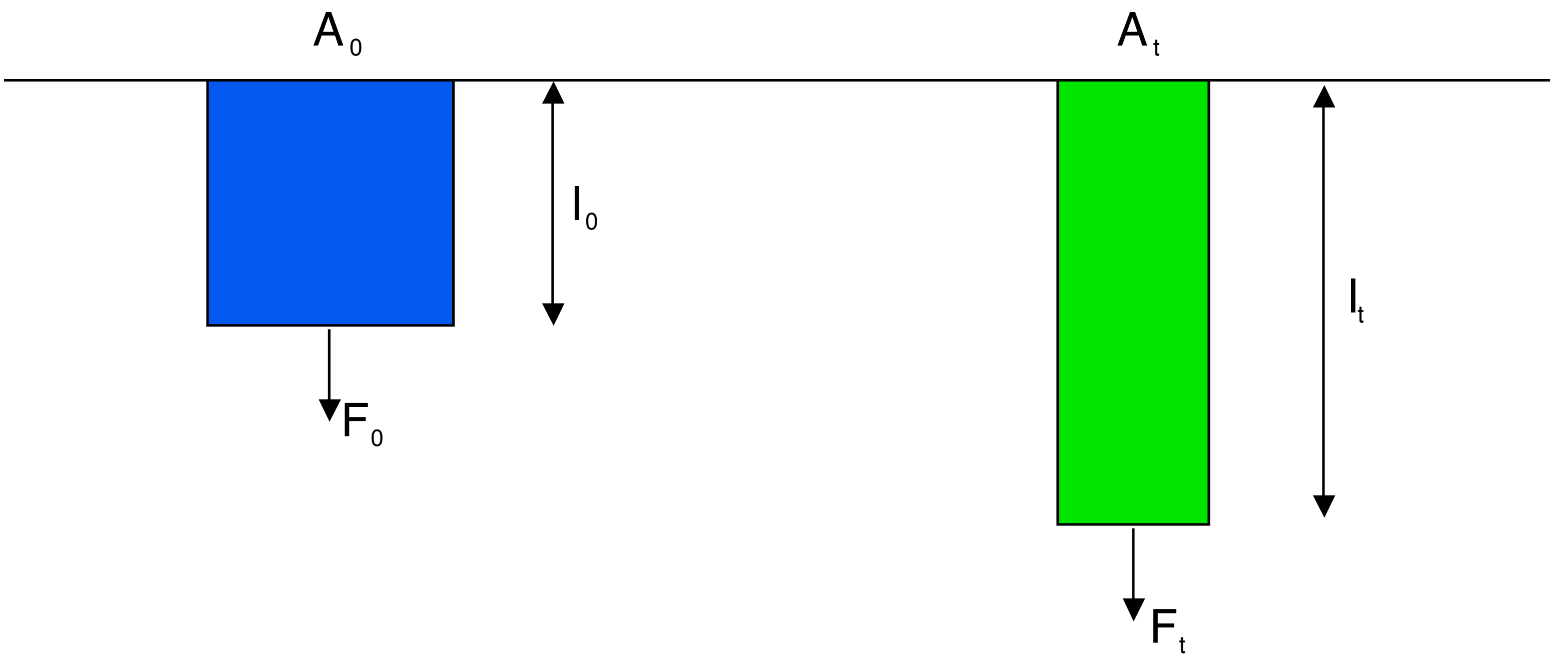


$$\det(F) = \lambda_1 \lambda_2 \lambda_3$$

$$S_{11} = \frac{\lambda_2 \lambda_3}{\lambda_1} \sigma_{11}$$

CVRTI

Example: Uniaxial Definition of Stress Tensors



$$\sigma = \frac{F_t}{A_t} \qquad T = \frac{F_t}{A_0} = \sigma \frac{A_t}{A_0} \qquad S = \frac{F_0}{A_0} = T \frac{l_0}{l_t} = \sigma \frac{A_t}{A_0} \frac{l_0}{l_t}$$

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Vectorial Representation of Stress Tensors

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \rightarrow \hat{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{13} & \sigma_{23} \end{pmatrix}^T$$

(Cauchy stress tensor)

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix} \rightarrow \hat{S} = \begin{pmatrix} S_{11} & S_{22} & S_{33} & S_{12} & S_{13} & S_{23} \end{pmatrix}^T$$

(2. Piola-Kirchhof stress tensor)

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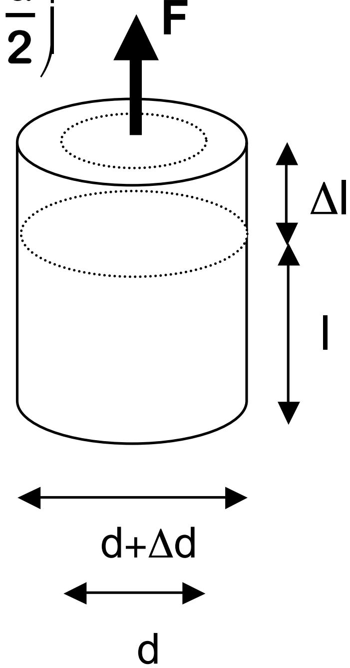
Young's Modulus and Poisson's Ratio

Deformable solid body

$$\Delta l = \frac{1}{E}\frac{l\,\mathbf{F}}{A} \qquad \frac{-\Delta d}{d}\frac{l}{\Delta l} = \nu$$

$$\mathbf{A} = \pi\left(\frac{\mathbf{d}}{\mathbf{2}}\right)^{2}$$



F: Force [N]

l: Length [m]

Δl: Change of length [m]

d: Diameter [m]

Δd: Change of diameter [m]

A: Area [m^2]

E: Young's modulus [N/m^2]

ν: Poisson's ration

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Vectorial Strain-Stress Relationship

$$\sigma_{ij} = C_{ijrs}\varepsilon_{rs}$$

σ: Cauchy stress tensor, 2nd order, symmetric

ε: Classical strain tensor, 2nd order, symmetric

C: Material/Elasticity tensor, 4th order, symmetric

$$\hat{\sigma} = C\hat{\varepsilon}$$

$\hat{\sigma}$: Cauchy stress vector, 6 dimensional

$\hat{\varepsilon}$: Classical strain vector, 6 dimensional

C: Material matrix, 6 x 6 dimensional, symmetric

CVRTI

Material Tensor in Matrix Representation

$$C = \begin{pmatrix} \begin{pmatrix} C_{1111} & C_{1112} & C_{1113} \\ C_{1121} & C_{1122} & C_{1123} \\ C_{1131} & C_{1132} & C_{1133} \end{pmatrix} & \begin{pmatrix} C_{1211} & C_{1212} & C_{1213} \\ C_{1221} & C_{1222} & C_{1223} \\ C_{1231} & C_{1232} & C_{1233} \end{pmatrix} & \begin{pmatrix} C_{1311} & C_{1312} & C_{1313} \\ C_{1321} & C_{1322} & C_{1323} \\ C_{1331} & C_{1332} & C_{1333} \end{pmatrix} \\ \begin{pmatrix} C_{2111} & C_{2112} & C_{2113} \\ C_{2121} & C_{2122} & C_{2123} \\ C_{2131} & C_{2132} & C_{2133} \end{pmatrix} & \begin{pmatrix} C_{2211} & C_{2212} & C_{2213} \\ C_{2221} & C_{2222} & C_{2223} \\ C_{2231} & C_{2232} & C_{2233} \end{pmatrix} & \begin{pmatrix} C_{2311} & C_{2312} & C_{2313} \\ C_{2321} & C_{2322} & C_{2323} \\ C_{2331} & C_{2332} & C_{2333} \end{pmatrix} \\ \begin{pmatrix} C_{3111} & C_{3112} & C_{3113} \\ C_{3121} & C_{3122} & C_{3123} \\ C_{3131} & C_{3132} & C_{3133} \end{pmatrix} & \begin{pmatrix} C_{3211} & C_{3212} & C_{3213} \\ C_{3221} & C_{3222} & C_{3223} \\ C_{3231} & C_{3232} & C_{3233} \end{pmatrix} & \begin{pmatrix} C_{3311} & C_{3312} & C_{3313} \\ C_{3321} & C_{3322} & C_{3323} \\ C_{3331} & C_{3332} & C_{3333} \end{pmatrix} \end{pmatrix}$$

Strain-Stress Relationship

$$S_{ij} = C_{ijrs} E_{rs}$$

S: 2nd Piola-Kirchhoff stress tensor, 2nd order, symmetric

E: Lagrangian strain tensor, 2nd order, symmetric

C: Material/Elasticity tensor, 4th order, symmetric

$$\hat{S} = C\hat{E}$$

$\hat{S}$: 2nd Piola-Kirchhoff stress vector, 6 dimensional

$\hat{E}$: Lagrangian strain vector, 6 dimensional

C: Material matrix, 6 x 6 dimensional, symmetric

Geometrical Nonlinear Elasticity: Large Strain

Description of elasticity for large deformations: $S = S(F)$

S: 2. Piola - Kirchhoff stress tensor

F: Deformation gradient tensor

Invariance concerning translation and rotation: $S = S(C)$ or $S = S(E)$

C: Cauchy - Green deformation tensor

E: Lagrangian strain tensor

For Isotropy: $S(C) = \sum_{i=0}^{n} a_i C^i$

Cayley – Hamilton – Theorem: $S(C) = a_0 I + a_1 C + a_2 C^2$

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Hyperelastic Material - Strain Energy Density Function

Definition: The 2. Piola-Kirchhoff stress tensor/vector is defined as derivative of a strain energy density function with respect to the components of the Lagrangian strain tensor/vector.

$$\hat{S} = \frac{\partial W}{\partial \hat{E}}$$

$\hat{S}$: 2nd Piola-Kirchhoff stress vector

W: Strain energy density function

$\hat{E}$: Lagrangian strain vector

$$C_0 = \frac{\partial \hat{S}}{\partial \hat{E}}$$

C_0: Incremental material tensor

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Strain Energy Density Function Defined with Invariants

$W(I_1, I_2) = a_{10}(I_1 - 3) + a_{01}(I_2 - 3)$ Rubber Mooney 40

W: Strain energy density function

$I_C = I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$

$II_C = I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2$

a_{10}, a_{01}: Material parameters

C: Cauchy strain tensor

$\lambda_1, \lambda_2, \lambda_3$: Eigenvalues

$$\hat{S} = \left(\frac{\partial W}{\partial \hat{E}_1} \quad \frac{\partial W}{\partial \hat{E}_2} \quad \frac{\partial W}{\partial \hat{E}_3} \quad \frac{\partial W}{\partial \hat{E}_4} \quad \frac{\partial W}{\partial \hat{E}_5} \quad \frac{\partial W}{\partial \hat{E}_6} \right)^T$$

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Elastical Modeling of Left Ventricle

Strain energy function	Species	Model	Type
$W = \frac{\mu}{k}\left(\lambda_1^{\,k} + \lambda_2^{\,k} + \lambda_3^{\,k} - 3\right)$	Rabbit	Needleman 83	polynomial
$W = \frac{\beta}{2\alpha}\left(e^{\alpha(I_1-3)} - 1\right)$	Dog	Demiray 76	exponential
$W = \frac{C}{\alpha}e^{\alpha\left(\sqrt{I_3}+\frac{nI_2}{I_3}+\frac{(n-1)I_1}{2}-3n+\frac{1}{2}\right)}$	Jack Rabbit	Yang 91	exponential
$W = \frac{E}{\nu\beta^2}\left(\frac{\nu}{\nu+1}\sum_{i=1}^{3}e^{\beta E_i} + \frac{1-2\nu}{1+\nu}e^{-\frac{\nu\beta}{1-2\nu}\sum_{i=1}^{3}E_i}\right)$	Rat	Janz 74	exponential

W: Strain energy density function

λ: Principal stretches

I: Invariants of Lagrangian strain tensor

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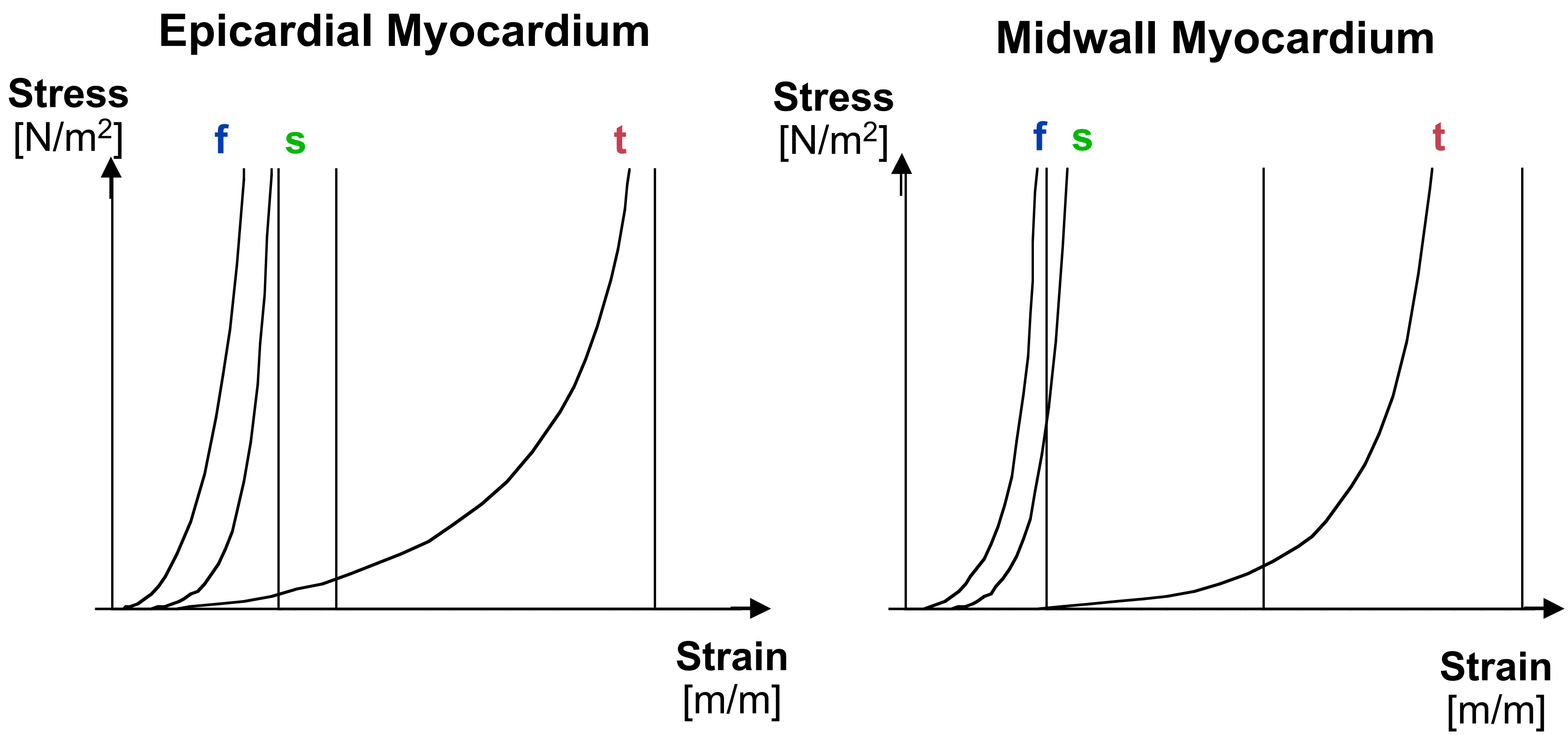
Inhomogeneous, Anistropic Strain-Stress Relationship



f: Fiber direction s: Sheet direction t: Sheet normal

(Hunter et al., Computational Biology, 1995)

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Strain Energy Density Function for Myocard (Hunter 95)

$$W = \iiint_V k_1 \frac{E_{11}^2}{\left(a_1 - |E_{11}|\right)^{\beta_1}} + k_2 \frac{E_{22}^2}{\left(a_2 - |E_{22}|\right)^{\beta_2}} + k_3 \frac{E_{33}^2}{\left(a_3 - |E_{33}|\right)^{\beta_3}}$$

$$+ k_4 \frac{E_{12}^2}{\left(a_4 - |E_{12}|\right)^{\beta_4}} + k_5 \frac{E_{23}^2}{\left(a_5 - |E_{23}|\right)^{\beta_5}} + k_6 \frac{E_{31}^2}{\left(a_6 - |E_{31}|\right)^{\beta_6}} dV$$

W: Strain energy density function

E_{ij}: Coefficients of Lagrangian Strain Tensor

$k_1 \ldots k_6, a_1 \ldots a_6, \beta_1 \ldots \beta_6$: Constants

Finite element method with Gaussian quadrature

CVRTI

Group Work

Specify 10 materials in the class room with anisotropy of their physical properties. Which of those are transversely isotropic?